

Quantization rule solution to the Hulthén potential in arbitrary dimension by a new approximate scheme for the centrifugal term

Sameer M. Ikhdaire^{1,*}

¹*Physics Department, Near East University, Nicosia, North Cyprus, Turkey*

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Abstract

The bound state energies and wave functions for a particle exposed to the Hulthén potential field in the D -dimensional space are obtained within the improved quantization rule for any arbitrary l state. The present approximation scheme used to deal with the centrifugal term in the effective Hulthén potential is systematic and accurate. The solutions for the three-dimensional ($D = 3$) case and the s -wave ($l = 0$) case are briefly discussed.

Keywords: Hulthén potential, improved quantization rule, approximation schemes.

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*E-mail: sikhdaire@neu.edu.tr

I. INTRODUCTION

It is well-known that the exact analytical solution of the hyperradial Schrödinger equation, in any arbitrary spatial dimension ($D \geq 2$), for its bound state energy levels, is fundamental in understanding the bound energy spectrum of nonrelativistic and relativistic quantum mechanics, since the resulting wave function contains all the necessary information to describe fully any quantum system. There are only a few potentials for which the radial Schrödinger equation can be solved explicitly for all n and l quantum numbers. One of these exactly solvable potentials is the Hulthén potential [1,2] which can be solved in a closed form for s wave ($l = 0$). However, the three-dimensional radial Schrödinger equation for the spherically symmetric Hulthén potential cannot be solved analytically for $l \neq 0$ states because of the centrifugal term $\sim r^{-2}$ [3-5]. The Hulthén potential is one of the important molecular potentials used in different areas of physics to describe the interaction between two atoms and has attracted a great of interest for some decades in the history of quantum chemistry. Until now, it has also been used extensively to describe the molecular structure and a possible form of atomic and nuclear interactions [6-10]. Further, the study of this potential is only a special example of those exponential type potentials [11] such as the modified hyperbolic-type potentials (Scarf, modified Rosen-Morse, modified second type Pöschl-Teller) [12], Eckart [13], Rosen-Morse [14], Manning-Rosen [15] potentials and so forth. So far, numerous attempts have been developed to calculate the bound-state energies such as variational [10], supersymmetry quantum mechanics [16,17], Nikiforov-Uvarov method (NU) [5,14,18,19], asymptotic iteration method (AIM) [20], hypervirial perturbation [21], shifted $1/N$ expansion (SE) and modified shifted $1/N$ expansion (MSE) [22], exact (improved) quantization rule (EQR or IQR) [23,24], perturbative formalism [25-29], polynomial solution [30], wave function ansatz method [31], factorization method [32] and tridiagonal J-matrix representation (TJM) [33] which split original Hamiltonian into two parts as $H = H_0 + V$ where H_0 is the part of the Hamiltonian that could be treated analytically while the remaining part, V , has to be treated numerically. to solve the radial Schrödinger, Klein-Gordon and Dirac equations exactly or quasi-exactly for $l \neq 0$ within a given potential.

Recently, Ma and Xu have proposed an exact (improved) quantization rule (EQR or IQR) and shown its power in calculating the energy levels of all bound states for some solvable quantum systems [23,24]. The method has been shown to be effective for calculating

the bound state solutions of the Schrödinger and Dirac wave equations with a spherically symmetric potential [34-39]. So far, it has been applied, with great success, to study a great number of potentials like the rotating Morse [34,35], the Kratzer-type [36], the trigonometric Rosen-Morse [37], the hyperbolic and the second Pöschl-Teller-like potentials [38], the Hulthén potential [39] and the Woods-Saxon potential [40] and so forth. Very recently, Gu and Sun [39] have extended the application of the IQR to the solution of the D -dimensional Schrödinger equation with the Hulthén potential for $l \neq 0$ using the usual approximation to deal with the centrifugal term [41-43].

Very recently, Dong [12] has introduced a more beautiful exact quantization rule to simplify the calculation of the energy levels for exactly solvable quantum systems. The energy spectra of the modified hyperbolic-type potentials have been carried out by this rule. Qiang and Dong [44] have found a proper quantization rule (PQR) and showed that the previous complicated and tedious calculations for the energy spectra can be greatly simplified. This new quantization rule can be applied to any exactly solvable potential. Qiang-Dong PQR has been further applied to exactly solvable shape invariant potentials [45]. Very recently, Yin *et al.* [46] have shown that the SWKB is exact for all shape invariant potentials (SIPs).

In this paper, we aim to extend the study of Ref. [39] by using a new improved approximate scheme to deal with the centrifugal term. Further, we solve the present potential model on the assumption that the space may possess an arbitrary number of spatial dimensions D . This arbitrary dimensional study enables one to give analytical tests using energy calculations for interdimensional degeneracy, i.e., $(n, l, D) \rightarrow (n, l \pm 1, D \mp 2)$ corresponding to the confined $D = 2 - 4$ dimensional Hulthén potential.

It is worth noting that this alternative approximating approach has shown its accuracy in calculating the analytic and numerical energy spectrum of the Hulthén potential for $l \neq 0$ [3-5]. Further, it has also been applied to the spin and pseudospin symmetries, e.g., Wei and Dong have studied the approximation of the Dirac equation with scalar and vector modified and deformed generalized Pöschl-Teller and Manning-Rosen potentials within the improved approximation formula to the centrifugal term [47-50].

This paper is organized as follows. In Sec. 2, the EQR (IQR) method is reviewed and extended to any arbitrary dimension ($D \geq 2$). In Sec. 3, the D -dimensional ($D \geq 2$) Schrödinger equation is solved by this method with $l \neq 0$ quantum numbers to obtain the

energy eigenvalues. In Sec. 4, we calculate the corresponding hyperradial wave functions of the Hulthén potential. Finally, some conclusions are given in Sec. 5.

II. EXACT (IMPROVED) QUANTIZATION RULE

A brief outline to the improved quantization rule is presented with an extension to the D -dimensional space ($D \geq 2$). The details can be found in Refs. [23,24]. The IQR has recently been proposed to solve exactly the one-dimensional (1D) Schrödinger equation:

$$\psi''(x) + k(x)^2 \psi(x) = 0, \quad k(x) = \frac{\sqrt{2\mu[E - V(x)]}}{\hbar}, \quad (1)$$

where the prime denotes the derivative with respect to the variable x . Here μ is the reduced mass of the two interacting particles, $k(x)$ is the momentum and $V(x)$ is a piecewise continuous real potential function of x . The Schrödinger equation is equivalent to the Riccati equation

$$\phi'(x) + \phi(x)^2 + k(x)^2 = 0, \quad (2)$$

where $\phi(x) = \psi'(x)/\psi(x)$ is the logarithmic derivative of wave function $\psi(x)$. Due to the Sturm-Liouville theorem, the $\phi(x)$ decreases monotonically with respect to x between two turning points, where $E \geq V(x)$. Specifically, as x increases across a node of the wave function $\psi(x)$, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again.

Moreover, Ma and Xu [23,24] have generalized this exact quantization rule to the three-dimensional (3D) radial Schrödinger equation with spherically symmetric potential by simply making the replacements $x \rightarrow r$ and $V(x) \rightarrow V_{\text{eff}}(r)$:

$$\int_{r_A}^{r_B} k(r) dr = N\pi + \int_{r_A}^{r_B} k'(r) \frac{\phi(r)}{\phi'(r)} dr, \quad k(r) = \frac{\sqrt{2\mu[E_{n,l} - V_{\text{eff}}(r)]}}{\hbar}, \quad (3)$$

where r_A and r_B are two turning points determined from the relation $E_{n,l} = V_{\text{eff}}(r)$, $N = n+1$ is the number of nodes of $\phi(r)$ in the region $E_{n,l} \geq V_{\text{eff}}(r)$ and it is larger by one than the number of nodes of wave function $\psi(r)$. The first term $N\pi$ is the contribution from the nodes of the logarithmic derivative of wave function, and the second term in (3) is called the quantum correction. It is found that, for all well-known exactly solvable quantum systems, this quantum correction is independent of the number of nodes of wave function

of the system. This means that it is enough to consider the ground state in calculating the quantum correction, i.e.,

$$Q_c = \int_{r_A}^{r_B} k_0'(r) \frac{\phi_0(r)}{\phi_0'(r)} dr = \pi q, \quad (4)$$

The quantization rule still holds for Schrödinger equation with spherically symmetric potential in D dimensions. In what follows, we shall employ this method to extend the work of Ref. [39] by using an improved approximation to the centrifugal term.

III. EIGENVALUES OF THE HULTHÉN POTENTIAL

The Schrödinger equation with spherically symmetric potential $V(r)$ for $l \neq 0$ takes the simple form

$$\left(-\frac{\hbar^2}{2\mu} \nabla_D^2 + V(r) - E_{n,l} \right) \psi_{n,l,m}(r, \Omega_D) = 0, \quad (5)$$

where the representation of the Laplacian operator ∇_D^2 , in spherical coordinates, is

$$\nabla_D^2 = \frac{\partial^2}{\partial r^2} + \frac{(D-1)}{r} \frac{\partial}{\partial r} - \frac{l(l+D-2)}{r^2}, \quad (6)$$

and

$$\psi_{n,l,m}(r, \Omega_D) = \psi_{n,l}(r) Y_l^m(\Omega_D), \quad \psi_{n,l}(r) = r^{-(D-1)/2} R(r), \quad (7)$$

where $Y_l^m(\Omega_D)$ is the hyperspherical harmonics. The wave functions $\psi_{n,l,m}(r, \Omega_D)$ belong to the energy eigenvalues $E_{n,l}$ and $V(r)$ stands for the Hulthén potential in the configuration space and r represents the D -dimensional intermolecular distance $\left(\sum_{i=1}^D x_i^2 \right)^{1/2}$.

Further, substituting Eqs. (6) and (7) into Eq. (5) yields the wave equation satisfying the radial wave function $R(r)$ in a simple analogy to the $2D$ and $3D$ radial Schrödinger equation

$$R''(r) + \frac{2\mu}{\hbar^2} [E_{n,l} - V_{eff}(r)] R(r) = 0, \quad (8)$$

where $V_{eff}(r)$ is the Hulthén effective potential in D dimensions defined by

$$V_{eff}(r) = -Ze^2\alpha \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{(\Lambda^2 - 1)\hbar^2}{8\mu r^2}, \quad (9)$$

with the parameter

$$\Lambda = 2l + D - 2. \quad (10)$$

The radial wave function $R(r)$ satisfying Eq. (8) should be normalizable and finite near $r = 0$ and $r \rightarrow \infty$ for the bound-state solutions. The wave equation (8) with the Hulthén potential is an exactly solvable problem for $l = 0$ (s -wave) [2,51-53], however, it cannot be solved analytically when $l \neq 0$ because of the centrifugal barrier term, i.e., $(\Lambda^2 - 1) \hbar^2 r^{-2} / (8\mu)$. Therefore, to solve Eq. (8) analytically, we must use a new approximation scheme of the exponential-type proposed recently by Jia *et al* (cf. e.g., [54-59]) to deal with the centrifugal term:

$$\frac{1}{r^2} \approx \alpha^2 \left(c_0 + \frac{e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \right), \quad (11)$$

where the dimensionless constant $c_0 = 1/12$ is exact as reported by other authors (cf. e.g., [3-5]). Very recently, we have applied the above approximation scheme (11) to obtain improved bound state solutions to the Schrödinger equation with the Manning-Rosen potential for arbitrary l -waves [15]. Obviously, the above approximation to the centrifugal term turns to r^{-2} when the parameter α goes to zero (small screening parameter α) as

$$\lim_{\alpha \rightarrow 0} \left[\alpha^2 \left(c_0 + \frac{1}{e^{\alpha r} - 1} + \frac{1}{(e^{\alpha r} - 1)^2} \right) \right] = \frac{1}{r^2}, \quad (11a)$$

which shows that the usual approximation is the limit of our approximation (cf. e.g., [4] and the references therein). Further, by defining

$$a = Ze^2\alpha, \quad b = \alpha^2 L^2, \quad L^2 = \frac{\hbar^2}{2\mu} \left(l + \frac{D-1}{2} \right) \left(l + \frac{D-3}{2} \right), \quad (12)$$

then we have from Eq. (8):

$$R''(r) + \frac{2\mu}{\hbar^2} \left[E_{n,l} + a \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} - b \left(c_0 + \frac{e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \right) \right] R(r) = 0, \quad (13)$$

where $E_{n,l}$ is the bound state energy of the system and n and l signify the radial and angular quantum numbers, respectively.

We now study this system through the improved exact quantization rule. At first, we introduce a new variable

$$z(r) = \frac{e^{-\alpha r}}{1 - e^{-\alpha r}}, \quad z'(r) = -\alpha z(1 + z), \quad (14)$$

where $r \in (0, \infty)$ and $z \in (0, \infty)$. Overmore, the turning points z_A and z_B are determined by solving $V_{eff}(z) = bz^2 + (b - a)z + bc_0 = E_{n,l}$ as follows:

$$z_A = \frac{a}{2b} - \frac{1}{2} - \frac{1}{2b} \sqrt{(a - b)^2 + 4b(E_{n,l} - bc_0)}, \quad z_B = \frac{a}{2b} - \frac{1}{2} + \frac{1}{2b} \sqrt{(a - b)^2 + 4b(E_{n,l} - bc_0)}, \quad (15)$$

with the properties

$$z_A + z_B = \frac{a}{b} - 1, \quad z_A z_B = -\frac{E_{nl}}{b} + c_0. \quad (16)$$

The momentum $k(z)$ between two turning points is expressed as

$$k(z) = \frac{\sqrt{2\mu b}}{\hbar} \sqrt{-z^2 + \frac{(a-b)}{b}z - \frac{(bc_0 - E_{n,l})}{b}} = \frac{\sqrt{2\mu b}}{\hbar} \sqrt{(z_B - z)(z - z_A)}, \quad (17a)$$

$$\frac{dk(z)}{dz} = \frac{\sqrt{2\mu b}}{2\hbar} \left(\sqrt{\frac{z_B - z}{z - z_A}} - \sqrt{\frac{z - z_A}{z_B - z}} \right), \quad (17b)$$

The Riccati equation (2) now becomes

$$-\alpha z(z+1) \frac{d\phi_0(z)}{dz} = -\frac{2\mu}{\hbar^2} [E_0 - bz^2 + (a-b)z - bc_0] - \phi_0(z)^2, \quad (18)$$

having the only possible solution satisfying

$$\phi_0(r) = c_1 z + c_2, \quad \phi_0'(r) = -\alpha c_1 z(1+z), \quad c_1 > 0. \quad (19)$$

where we have used $\phi_0(r) \equiv \phi_0(z)$. Substituting $\phi_0(z)$ into Eq. (18), one has the ground state energy eigenvalue and wave function solutions

$$\begin{cases} \phi_0(z) = \nu \alpha z + \alpha \left(\frac{\mu Z e^2}{\hbar^2} \frac{1}{\nu \alpha} - \frac{\nu}{2} \right), \\ \nu = \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\mu}{\hbar^2} L^2} = l + \frac{D-1}{2}, \quad \nu \geq 1, \\ \tilde{E}_{n=0} = E_0 - bc_0 = -\frac{\hbar^2 \alpha^2}{2\mu} \left(\frac{\mu Z e^2}{\hbar^2} \frac{1}{\nu \alpha} - \frac{\nu}{2} \right)^2. \end{cases} \quad (20)$$

After a lengthy algebra but straightforward, we can calculate the integral of the quantum correction (4) based on the ground state as

$$Q_c = \pi q = \pi \left(\frac{\sqrt{2\mu}}{\hbar} L + \nu - 1 \right). \quad (21)$$

The integral of the momentum $k(r)$ in the quantization rule (3) is calculated as

$$\begin{aligned} \int_{r_A}^{r_B} k(r) dr &= -\frac{\sqrt{2\mu b}}{\alpha \hbar} \int_{z_A}^{z_B} \left(\frac{\sqrt{(z - z_A)(z_B - z)}}{z} - \frac{\sqrt{(z - z_A)(z_B - z)}}{1+z} \right) dz \\ &= \frac{2\mu}{\hbar^2} L \left(1 + \sqrt{c_0 - \frac{E_{n,l}}{b}} - \sqrt{c_0 + \frac{(a - E_{n,l})}{b}} \right) \pi. \end{aligned} \quad (22)$$

Using the relations (21) and (22), the improved quantization rule (3) turn out to be

$$\pi \frac{\sqrt{2\mu}}{\hbar} L \left(1 + \sqrt{c_0 - \frac{E_{n,l}}{b}} - \sqrt{c_0 + \frac{(a - E_{n,l})}{b}} \right) = \pi \left(\frac{\sqrt{2\mu}}{\hbar} \Lambda + n + \nu \right). \quad (23)$$

Thus, one can finally find the approximation to the bound state energy levels E_{nl} for the D -dimensional Hulthén potential,

$$E_{n,l}^{(D)} = \frac{\hbar^2 \alpha^2}{2\mu} \left\{ \left(l + \frac{D-1}{2} \right) \left(l + \frac{D-3}{2} \right) c_0 - \left[\frac{\mu Z e^2}{\hbar^2 (n+l+\frac{D-1}{2}) \alpha} - \frac{(n+l+\frac{D-1}{2})}{2} \right]^2 \right\}, \quad (24)$$

where $n, l = 0, 1, 2, \dots$. Therefore, the energy spectrum in $3D$ space can be obtained as

$$E_{n,l} = \frac{\hbar^2 \alpha^2}{2\mu} \left\{ \frac{l(l+1)}{12} - \left[\frac{\mu Z e^2}{\hbar^2 (n+l+1) \alpha} - \frac{(n+l+1)}{2} \right]^2 \right\}, \quad n, l = 0, 1, 2, \dots, \quad (25)$$

which is identical to Eq. (34) of Ref. [4]. In the case of the s -wave ($l = 0$), the previous relation turns out to become

$$E_n = -\frac{\hbar^2 \alpha^2}{2\mu} \left[\frac{\mu Z e^2}{\hbar^2 (n+1) \alpha} - \frac{(n+1)}{2} \right]^2, \quad n = 0, 1, 2, \dots, \quad (26)$$

which is identical to the ones obtained before using the AIM [20], SUSYQM approach [60-64], quasi-linearization method [65] and NU method [42,66]. Overmore, if we take the dimensionless constant $c_0 = 0$ in the present approximation, Eq. (24) reduces to

$$E_{n,l} = -\frac{\hbar^2 \alpha^2}{2\mu} \left[\frac{\mu Z e^2}{\hbar^2 (n+l+\frac{D-1}{2}) \alpha} - \frac{(n+l+\frac{D-1}{2})}{2} \right]^2, \quad (27)$$

which is consistent with the energy eigenvalues formula given in Eq. (32) of Ref. [20], Eq. (24) of Ref. [67] and Eq. (28) of Ref. [42] when $D = 3$. By taking the chosen parameters $\hbar = 2\mu = e = 1$ and for $Z = 1$, the above result is consistent with Eq. (24) of Ref. [39]. The critical screening parameter can be found as $\alpha_c = \frac{2\mu Z e^2}{\hbar^2 (n+l+1)^2}$ when $E_{nl} = 0$ and $c_0 = 0$.

IV. EIGENFUNCTIONS

We are now in the position to study the corresponding eigenfunction of this quantum system for completeness. The Riccati equation of the relation (8) is [68]

$$\phi'(r) = -\frac{2\mu}{\hbar^2} [E_{nl} - V_{eff}(r)] - \phi(r)^2, \quad (28)$$

where

$$\phi(r) = \frac{R'(r)}{R(r)}. \quad (29)$$

Based on

$$R(r) = e^{\int^r \phi(r) dr} = e^{-\frac{1}{\alpha} \int^r \frac{1}{z(z+1)} \phi(z) dz}, \quad (30)$$

and using Eq. (19), we can easily calculate the eigenfunction of the ground state as

$$R_0(r) = N_0 (e^{-\alpha r})^{\tilde{\varepsilon}_0} (1 - e^{-\alpha r})^\nu, \quad \tilde{\varepsilon}_0 > 0, \quad \nu \geq 1, \quad (31)$$

where

$$\tilde{\varepsilon}_{n=0} = \sqrt{\frac{2\mu}{\hbar^2 \alpha^2} (bc_0 - E_0)} = \frac{\mu Z e^2}{\hbar^2} \frac{1}{\nu \alpha} - \frac{\nu}{2}, \quad (32)$$

with ν is defined in Eq. (20) and N_0 is the normalization constant.

Let us find the eigenfunction for any quantum number n . At first, considering the boundary conditions

$$y = \begin{cases} 0 & \text{when } r \rightarrow \infty, \\ 1 & \text{when } r \rightarrow 0, \end{cases} \quad (33)$$

with $R(y) \rightarrow 0$, based on Eq. (31), we may define a more general radial eigenfunctions, valid for any quantum number n , of the form:

$$R(y) = y^{\tilde{\varepsilon}_{n,l}} (1 - y)^\nu F(y), \quad y = e^{-\alpha r}, \quad \tilde{\varepsilon}_{n,l} > 0, \quad \nu \geq 1, \quad (34)$$

satisfying the boundary conditions in Eq. (33), where

$$\tilde{\varepsilon}_{n,l} = \frac{\mu Z e^2}{\hbar^2} \frac{1}{(n + \nu) \alpha} - \frac{n + \nu}{2} > 0. \quad (35)$$

Substituting Eq. (34) into Eq. (8) leads to the following hypergeometric equation

$$y(1 - y) F''(y) + [1 + 2\tilde{\varepsilon}_{n,l} - (1 + 2\tilde{\varepsilon}_{n,l} + 2\nu)y] F'(y) - \left[\nu(\nu + 2\tilde{\varepsilon}_{n,l}) - \frac{2\mu Z e^2}{\hbar^2 \alpha} \right] F(y) = 0, \quad (36)$$

whose solutions are the hypergeometric functions

$$F(y) = {}_2F_1(A, B; C; y) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(B)} \sum_{k=0}^{\infty} \frac{\Gamma(A+k)\Gamma(B+k)}{\Gamma(C+k)} \frac{y^k}{k!}, \quad (37)$$

where

$$\begin{aligned} A &= \tilde{\varepsilon}_{n,l} + \nu - \sqrt{\tilde{\varepsilon}_{n,l}^2 + \frac{2\mu Z e^2}{\hbar^2 \alpha}} = -n, \\ B &= \tilde{\varepsilon}_{n,l} + \nu + \sqrt{\tilde{\varepsilon}_{n,l}^2 + \frac{2\mu Z e^2}{\hbar^2 \alpha}}, \\ C &= 1 + 2\tilde{\varepsilon}_{n,l}. \end{aligned} \quad (38)$$

By considering the finiteness of the solutions, the quantum condition is given by

$$\tilde{\varepsilon}_{n,l} + \nu - \sqrt{\tilde{\varepsilon}_{n,l}^2 + \frac{2\mu Ze^2}{\hbar^2 \alpha}} = -n, \quad n = 0, 1, 2, \dots, \quad (39)$$

from which we obtain Eq. (25). Now, we may write down the radial wave functions (34) as

$$R(r) = \mathcal{N}_{nl} (e^{-\alpha r})^{\tilde{\varepsilon}_{n,l}} (1 - e^{-\alpha r})^\nu {}_2F_1(-n, n + 2(\tilde{\varepsilon}_{n,l} + \nu); 1 + 2\tilde{\varepsilon}_{n,l}; e^{-\alpha r}). \quad (40)$$

If we set $n = 0$ in Eq. (40), then we can easily obtain Eq. (31). Finally, the unnormalized total wave functions are obtained as

$$\psi_{n,l,m}(r, \Omega_D) = \mathcal{N}_{nl} r^{-(D-1)/2} (e^{-\alpha r})^{\tilde{\varepsilon}_{n,l}} (1 - e^{-\alpha r})^\nu {}_2F_1(-n, n + 2(\tilde{\varepsilon}_{n,l} + \nu); 1 + 2\tilde{\varepsilon}_{n,l}; e^{-\alpha r}) Y_l^m(\Omega_D). \quad (41)$$

which is identical to Eq. (42) of Ref. [4] when $D = 3$. Thus, the Jacobi polynomials can be expressed in terms of the hypergeometric functions [69]

$$P_n^{(A,B)}(1 - 2x) = \frac{\Gamma(n + 1 + A)}{n! \Gamma(1 + A)} {}_2F_1(-n, n + A + B + 1; A + 1; x). \quad (42)$$

The hypergeometric function ${}_2F_1(A, B; C; x)$ is a special case of the generalized hypergeometric function [69,70]

$${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \dots (\alpha_p)_k}{(\beta_1)_k (\beta_2)_k \dots (\beta_q)_k} \frac{x^k}{k!}, \quad (43)$$

where the Pochhammer symbol is defined by $(y)_k = \Gamma(y + k)/\Gamma(y)$.

Let us find the normalization constant. Introducing the change of parameters $y(r) = e^{-\alpha r}$ and making use of Eq. (41), with the help of Eq. (42), we are able to express the normalization condition $\int_0^\infty R(r)^2 dr = 1$ as

$$\frac{\mathcal{N}_{nl}^2}{\alpha} \int_0^1 y^{2\tilde{\varepsilon}_{n,l}-1} (1 - y)^{2l+D-1} [P_n^{(2\varepsilon_{nl}, 2l+D-2)}(1 - 2y)]^2 dy = 1. \quad (44)$$

Unfortunately, there is no formula available to calculate this key integration. Nevertheless, we can find the explicit normalization constant N_{nl} . For this purpose, it is not difficult to obtain the results of the above integral by using the following formulas [69-71],

$$P_n^{(\alpha, \beta)}(x) = (n + \alpha)! (n + \beta)! \sum_{p=0}^n \frac{1}{p! (n + \alpha - p)! (\beta + p)! (n + p)!} \left(\frac{x - 1}{2} \right)^{n-p} \left(\frac{x + 1}{2} \right)^p, \quad (45)$$

and

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \text{Re}(x), \text{Re}(y) > 0. \quad (46)$$

Thus, the normalization constant \mathcal{N}_{nl} is now obtained as

$$\mathcal{N}_{nl} = \frac{1}{(n+2l+D-2)!\Gamma(2\tilde{\varepsilon}_{n,l}+n+1)} \sqrt{\frac{\alpha\Gamma(2\tilde{\varepsilon}_{n,l}+2n+2l+D+1)}{\Gamma(2\tilde{\varepsilon}_{n,l}+2n+1) \sum_{p,q=0}^n (f_p f_q f_{p,q})^{-1}}}, \quad (47)$$

where

$$f_p = (-1)^p p! \Gamma(2\tilde{\varepsilon}_{n,l}+n-p+1) (2l+p+D-2)! (n+p)!, \quad (48a)$$

$$f_q = (-1)^q q! \Gamma(2\varepsilon_{n,l}+n-q+1) (2l+D+q-2)! (n+q)!, \quad (48b)$$

$$f_{p,q} = (2l+p+q+D-1)!. \quad (48c)$$

It is worth noting that one of the disadvantages of the EQR (IQR) approach is that it cannot get the eigenfunctions of studied potential models. The estimation shown in this section is only the reverse of logarithmic derivative to the original Schrödinger equation, i.e., $\phi_0(r) = \frac{d}{dr} \ln(R_0(r))$ [see Eq. (30)] [72]. The traditional method is thus used again.

V. CONCLUSIONS

In this work, we have applied an alternative method to obtain approximate energy eigenvalues and eigenfunctions of the D -dimensional Schrödinger equation for the Hulthén potential with $l \neq 0$ within the improved approximation scheme for the centrifugal term. The advantage of this method is that it gives the eigenvalues through the calculation of two integrations (21) and (22) and solving the resulting algebraic equation. First, we can easily obtain the quantum correction by only considering the solution of the ground state of quantum system since it is independent of the number of nodes of wave function for exactly solvable quantum system. Second, the wave functions have also been obtained by solving the Riccati equation. The general expressions obtained for the energy eigenvalues and wave functions can be easily reduced to the three-dimensional space ($D = 3$), s -wave ($l = 0$), the $c_0 = 0$ (usual approximation) cases. The method presented here is a systematic one, simple, practical and powerful than the other known methods. It is worth to extend this method to the solutions of other nonrelativistic [23,24,34,36-39] and relativistic [35] wave equations with different potential fields. Finally, it can be also used to deal with many exactly solvable quantum systems with wide range of potentials as stated by many authors [23,24,34-39].

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Appendix A: Integral Formulas

The following integral formulas are useful during the calculation of the momentum integral and the quantum correction terms [34,36,68]:

$$\int_{r_A}^{r_B} \frac{r}{\sqrt{(r-r_A)(r_B-r)}} dr = \frac{\pi}{2}(r_A + r_B), \quad (49)$$

$$\int_{r_A}^{r_B} \frac{1}{r\sqrt{(r-r_A)(r_B-r)}} dr = \frac{\pi}{\sqrt{r_A r_B}}, \quad (50)$$

$$\int_{r_A}^{r_B} \frac{1}{\sqrt{(r-r_A)(r_B-r)}} dr = \pi, \quad (51)$$

$$\int_{r_A}^{r_B} \frac{1}{r} \sqrt{(r-r_A)(r_B-r)} dr = \pi \left[\frac{1}{2}(r_A + r_B) - \sqrt{r_A r_B} \right], \quad (52)$$

$$\int_{r_A}^{r_B} \frac{1}{(a+br)\sqrt{(r-r_A)(r_B-r)}} dr = \frac{\pi}{\sqrt{(a+br_A)(a+br_B)}}, \quad r_B > r_A > 0. \quad (53)$$

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